Properties of Lamb Waves Relevant to the Ultrasonic Inspection of Thin Plates*

T. N. GRIGSBY†, MEMBER, IRE, AND E. J. TAJCHMAN†

Summary—**An experiment is described, using 1/32-inch thick steel plates, in which the insertion loss of artificial flaws was measured. The artificial flaws consisted of 1/32-inch wide saw cuts of depths varying from 0.000 inch to 0.021 inch in increments of 0.003 inch. No simple relationship was found relating the insertion loss to the depth of the saw cuts, although the presence of a flaw was usually indicated. Measurements of the group velocity of Lamb waves are described and the values obtained are shown to agree with calculated values. The theory of Lamb waves is discussed, and the results of computation of the frequency equation for the symmetrical case for a ratio of longitudinal velocity to shear velocity of 1.8 are given in the form of curves of phase velocity vs frequency-thickness product. Curves of group velocity vs frequency-thickness product for this case are also given. A formula is derived which gives group velocity as a function of phase velocity and frequency-thickness product. A derivation of the frequency equation is given.**

INTRODUCTION

FOR the purpose of nondestructive testing of thin plates, with thickness of the same order of magnitude as the wavelength, it is desirable that as much knowledge as possible be acquired about the possible modes of ultrasonic propagation in such plates. Although techniques of using Lamb waves for ultrasonic inspection were patented in 1951,[1] and although they have been applied in some cases,[2] they have failed to gain widespread acceptance. It is the purpose of this paper to describe some of the properties of Lamb wave propagation which are relevant to possible nondestructive testing applications.

The case of waves of ultrasound being guided by a thin elastic plate is, in general, qualitatively analogous to the case of electromagnetic waves being guided by a rectangular wave guide. In the case of ultrasound being guided by a thin plate, the modes of propagation are solutions to the equations of motion subject to the boundary conditions imposed by the medium in contact with the faces of the plate. In the corresponding case of electromagnetic waves being guided by a rectangular waveguide, the modes of propagation are solutions of Maxwell's equations subject to the boundary conditions imposed by the walls of the guide. Lamb waves include only those modes of vibration of an infinite plate in which the particle motion is independent of the direction which is parallel to the free surfaces and normal to the direction of propagation. However, in addition to Lamb waves, there are other modes which have particle motion only in the direction which is parallel to the free surfaces and normal to the direction of propagation. For the first and higher modes of this latter type, the frequency equation is identical to the frequency equation for the propagation of electromagnetic waves in a rectangular wave guide.[3] The nondispersive zeroth order mode of this case does not have a counterpart in the rectangular waveguide.[4] No attempt has been made so far to use these modes in nondestructive testing.

From the standpoint of nondestructive testing, the main features that distinguish Lamb waves from longitudinal and shear waves are that Lamb waves are dispersive and that there is an infinite number of possible modes.

LAMB WAVE PROPAGATION

Lamb waves are waves which can exist in an isotropic, lossless, elastic plate of infinite length and width, but of finite thickness, bounded on both of its faces by vacuum and which are independent of a direction parallel to the free surfaces and normal to the direction of propagation.

Several theoretical treatments of Lamb waves may be found in the literature.[5-11] The frequency equation relating phase velocity to frequency can be obtained by making use of Hooke's Law, Newton's second law of motion, and by satisfying the boundary conditions that

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† Denver Res. Inst., University of Denver, Denver, Colo.

[1] F. A. Firestone and D. S. Ling, "Methods and Means of Generating and Utilizing Vibrational Waves in Plates," U. S. Patent No. 2,536,128; 1951.

[2] D. C. Worlton, "Ultrasonic testing with Lamb waves," *Nondestructive Testing*, vol. 15, pp. 218–222; July–August, 1957.

[3] W. M. Ewing, W. S. Jardetzky, and F. Press, "Elastic Waves in Layered Media," McGraw-Hill Book Co., Inc., New York, N. Y., pp. 294–295; 1957.

[4] A. H. Meitzler, "Ultrasonic delay lines using shear modes in strips," IRE TRANS. ON ULTRASONIC ENGINEERING, vol. UE-7, p. 37; June, 1960. See especially, p. 37.

[5] H. Lamb, "On waves in an elastic plate," *Proc. Roy. Soc.* (*London*), vol. 93, pp. 114–128; March, 1917.

[6] F. A. Firestone and D. S. Ling, "Report on the Propagation of Waves in Plates—Lamb and Rayleigh Waves," Sperry Products Inc., Danbury, Conn., Tech. Rept. No. 50-6001; 1945.

[7] R. D. Mindlin, "An Introduction to the Mathematical Theory of Vibrations of Elastic Plates," U. S. Army Signal Corps. Engrg. Labs., Ft. Monmouth, N. J., Signal Corps Contract DA-36-039-SC-56772, pp. 2.12-2.51; 1955.

[8] W. M. Ewing, W. S. Jardetzky, and F. Press, *op. cit.*, pp. 281–288.

[9] D. C. Worlton, "Lamb Waves at Ultrasonic Frequencies," Hanford Atomic Products Operation, Richland, Wash., Rept. No. HW-60662, pp. 5–18; 1959.

[10] T. N. Grigsby and E. J. Tajchman, "Investigation of Elastic Waves in Thin Plates," U. S. Army Signal Corps Engrg. Labs., Ft. Monmouth, N. J., Interim Rept. on Ordnance, Contract DA-23-072-505-ORD-1, pp. 11–34; 1960.

[11] A. N. Holden, "Longitudinal modes of elastic waves in isotropic cylinders and slabs," *Bell Sys. Tech. J.*, vol. 30, pp. 956–969; October, 1951.

the tensional stress normal to the free surfaces and the shear stress tangential to the free surfaces must vanish. It can be shown that the frequency equation for Lamb waves may be split into two separate equations one of which applies to motions of the plate which are symmetrical about the medial plane and the other to motions which are anti-symmetrical about the medial plane.[8,10] For the convenience of the reader, this derivation appears in Appendix I.

The frequency equation for the symmetrical case is[12]

$$\frac{\tanh\left(\frac{\alpha h}{2}\right)}{\tanh\left(\frac{\beta h}{2}\right)} = \frac{(k^2+\beta^2)^2}{4k^2\alpha\beta}; \tag{1}$$

where

$$\alpha = k\sqrt{1-\frac{V^2}{V_L^2}},$$

$$\beta = k\sqrt{1-\frac{V^2}{V_S^2}},$$

in which

f = frequency
h = thickness of plate
V = Lamb wave phase velocity
k = wave number = $2\pi f/V$
V_L = velocity of longitudinal waves in an extended medium of the same elasticity and density as the plate
V_S = velocity of shear waves in an extended medium of the same elasticity and density as the plate.

The frequency equation for the antisymmetrical case is[13]

$$\frac{\tanh\left(\frac{\alpha h}{2}\right)}{\tanh\left(\frac{\beta h}{2}\right)} = \frac{4k^2\alpha\beta}{(k^2+\beta^2)^2}. \tag{2}$$

The frequency equations are solved by assuming a value of V and then computing the corresponding values of frequency-thickness product by successive approximations, preferably with the use of a digital computor. Worlton has found roots of these frequency equations for zirconium, stainless steel, uranium, brass and aluminum. and has plotted the results as curves of phase velocity vs the frequency-thickness product.[14] Firestone and Ling previously made calculations for aluminum and also gave curves of the group velocity. Normalization of the frequency equation permits the computation of a single set of curves for each of several values of the ratio of longitudinal velocity to shear velocity so that the desired solution to the frequency equation could readily be found from such curves for plates of any isotropic solid. A set of normalized curves showing phase velocity vs frequency-thickness product for a ratio of longitudinal velocity to shear velocity of 1.8 (*i.e.*, Poisson ratio = .277) is shown in Fig. 1.

It is easily shown (Appendix II) that the group velocity may be expressed in terms of phase velocity and frequency-thickness product, as follows:

$$V_g = V\left[1 - \frac{1}{1-(fh)\dfrac{dV}{d(fh)}}\right].$$

To normalize this expression with respect to shear wave velocity, let $V_{gn} = V_g/V_S$, $V_n = V/V_S$ and $(fh)_n = fh/V_S$; then,

$$V_{gn} = V_n\left[1 - \frac{1}{1-(fh)_n\dfrac{dV}{d(fh)_n}}\right].$$

which is the formula used to compute the curves of Fig. 2.

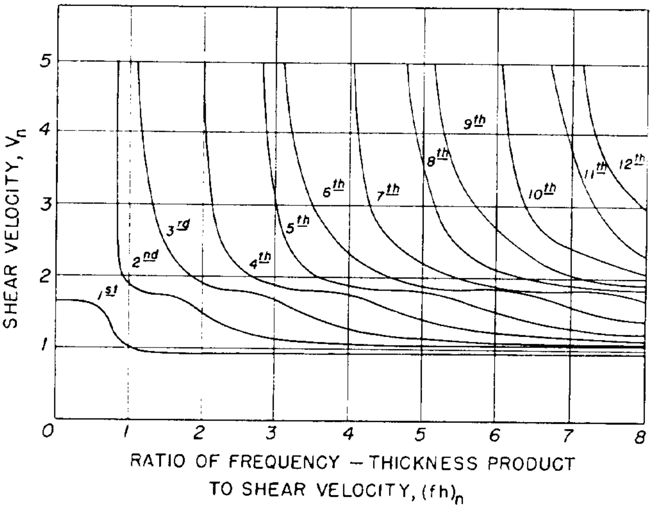

Fig. 1—Normalized curves of phase velocity vs frequency-thickness product for a ratio of longitudinal velocity to shear velocity of 1.8.

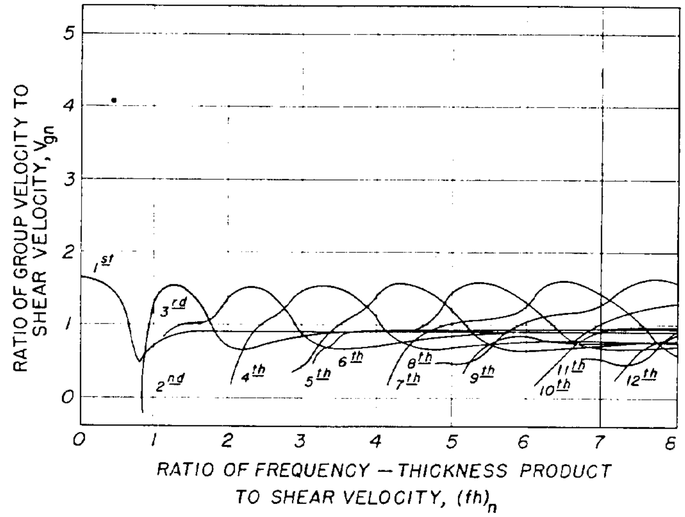

Fig. 2—Normalized curves of group velocity vs frequency-thickness product for a ratio of longitudinal velocity to shear velocity of 1.8.

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[12] H. Lamb, *op. cit.*, p. 116.

[13] *Ibid.*, p. 122.

[14] D. C. Worlton, "Lamb Waves at Ultrasonic Frequencies," Hanford Atomic Products Operation, Richland, Wash., Rept. No. HW-60662, pp. 31–32, pp. 51–58; 1959.

EXPERIMENTAL APPARATUS AND MATERIALS

A functional block diagram of the apparatus is shown in Fig. 3. The following equipment was used: gate pulse generator, Electropulse 4120A; pulsed oscillator, Arenberg PG-650-C; transducers, Branson type Z, 0.5 Mc, 1.6 Mc, 3.3 Mc, and 10 Mc, 0.5 inch in diameter; preamplifier, two Hewlett-Packard type 460A wide-band amplifiers in cascade; attenuator, Arenberg ATT 693; amplifier, Tektronix No. 121; oscilloscope, Tektronix 545 with 53/54C dual-trace preamplifier. The wavemeter was constructed using quartz crystals resonant at 0.5000 Mc, 1.6000 Mc, 3.3000 Mc, and 10.000 Mc.

The test plates consisted of precision-ground flat stock, Starrett type No. 496. Fig. 4 shows a 10 × 18-inch test plate; however, most of the data given below were obtained using 2 × 10-inch test plates, since no side reflections were observed for this size of plate. The test plates were supported in an aluminum frame so that they were bounded on both faces by air.

The positioning apparatus and lucite wedges used are shown in Fig. 4.

In order to generate Lamb waves of phase velocity, V, one can project a longitudinal wave onto the plate at an angle of incidence of

$$\theta = \sin^{-1}(V_1/V),$$

where V_1 is the velocity of the longitudinal wave.[15] For the data which follow, V_1 = velocity of longitudinal waves in lucite = 2680 m/sec.

The couplant used was ordinary motor oil.

MEASUREMENT OF GROUP VELOCITY

The method used for measuring group velocity was to set the transmitting and receiving transducers close together and measure the distance and delay time. The transmitting and receiving transducers were then moved further apart, and the distance and delay time were again measured. The difference between the two distances divided by the difference between the two delay times is the group velocity. There is an inherent uncertainty in the measurement of group velocity. Since Lamb waves are dispersive, the different components of the pulse travel with different velocities. Consequently, the transmitted and received pulses do not have the same shape (Fig. 5), and some arbitrary criterion must be established for deciding at what point in time a given pulse occurs. When the data of Table I were gathered, the time at which a given pulse occurred was taken to be the time at which its leading edge reached 50 per cent of the maximum voltage attained. The agreement between the calculated and experimental values of group velocity is good when one considers that the ratio of longitudinal to shear velocity in steel is actually 1.85 instead of 1.8, which was used in computing Fig. 2.

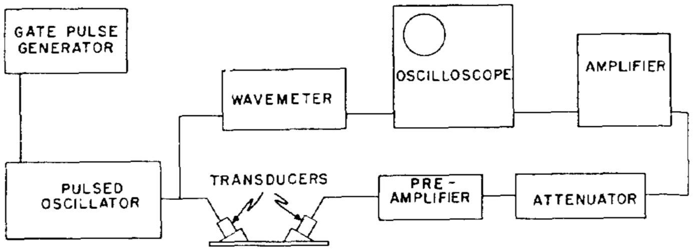

Fig. 3—Functional block diagram of apparatus.

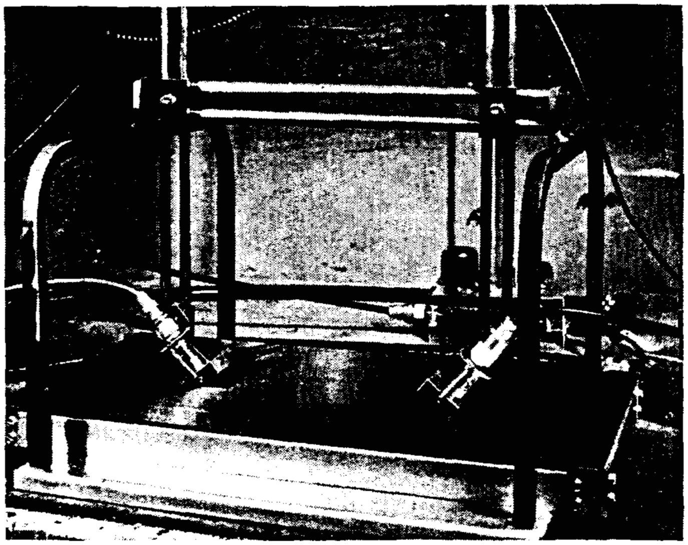

Fig. 4—Transducers, wedges, and positioning clamps.

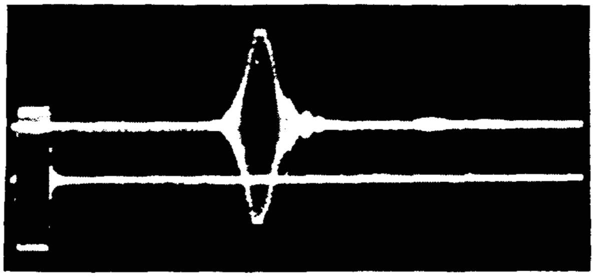

Fig. 5—Input waveform (lower trace, actually the output of the wavemeter of Fig. 3) and output waveform (upper trace) resulting from excitation of the second asymmetrical mode in a 1/32-inch thick steel plate at 2.46 Mc. The pulse duration is 15 μsec, and the total sweep time is 200 μsec.

TABLE I

Plate thickness, inches	Frequency, Mc.	Phase velocity, ips	Group velocity, ips	
			Experimental	Calculated (V_L/V_S = 1.8)
1/8	3.14	308,500	75,000	70,500
1/32	2.99	164,100	69,900	71,200
1/32	9.13	164,100	90,650	88,900
1/32	4.95	228,000	188,750	195,200
1/32	5.40	228,000	190,750	190,000
1/32	8.70	233,800	180,900	194,000

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[15] D. C. Worlton, "Ultrasonic testing with Lamb waves," *Nondestructive Testing*, vol. 15, p. 219; July–August, 1957.

MEASUREMENT OF INSERTION LOSS OF IDEALIZED FLAWS

In order to obtain data which might indicate possible applications of ultrasonic Lamb waves in the nondestructive testing of thin plates, an experiment was conducted using plates having highly idealized flaws. The primary objectives of this experiment were to determine whether or not such flaws would have significant insertion losses, and, if so, to attempt the correlation of insertion loss with flaw size.

The plates used in this experiment were 2 × 10 × .033-inches thick. The simulated flaws consisted of various depths of saw cuts, 1/32-inch wide. A typical plate with its saw cut is shown in Fig. 6. Nine such test plates were prepared from one 18 × 10 × 1/32-inch steel plate. Saw cuts which varied in depth from 0.003- to .021-inch in increments of 0.003 inch were made in seven of these plates.

The experiment performed with these plates consisted of comparing the transmission of Lamb waves in plates without saw cuts with the transmission in plates with saw cuts. The power level was read from the precision attenuator when it was set so that the signal at the input to the oscilloscope was 0.18 volts peak-to-peak. (If the pulse shape was distorted so that the top of the pulse was not flat, the 0.18 volts were measured across the average envelope of the pulse.) In order to minimize the experimental error, a standard technique was adopted for obtaining acoustic coupling from the lucite wedges to the plate. This procedure consisted of applying a spot of oil the size of a penny beneath each wedge, applying enough pressure so that some oil was squeezed from beneath each wedge and wiping off the excess in front of the wedges. The standard deviation of the experimental error was 3 db when measurements were being made on unslotted reference plates. Consequently, all measurements were repeated ten times and averaged so that the standard deviation of sample averages is approximately 1 db. In some instances, there was considerable distortion of the pulse which resulted in some additional error. The results of these measurements are presented in Table II.

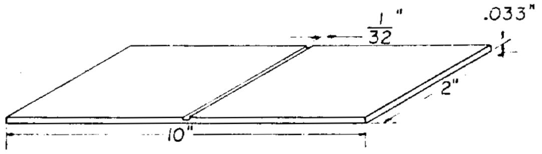

Fig. 6—Test plate with simulated flaw.

The angle given in Table II is the angle of incidence in lucite of the longitudinal wave used for excitation. The transmitting and receiving wedges were as nearly identical as it was possible to make them.

It is apparent from Table II that no simple relationship exists between the depth of the saw cut and the amount of insertion loss. It appears that ultrasonic inspection with Lamb waves is capable of indicating if there is a difference between a given plate and a reference plate. It does not appear that Lamb wave inspection will permit an unambiguous description of this difference.

The authors are not able to give a quantitative explanation of the results of these saw-cut measurements at this time. A qualitative explanation is as follows: The energy of the wave incident upon the saw cut is concentrated in a single mode which is determined by the frequency-thickness product of the plate and the transmitting wedge used. When the incident wave reaches the first thickness discontinuity, mode conversion will, in general, take place so that all boundary conditions at the discontinuity will be satisfied. When that part of the wave which is transmitted beneath the saw cut reaches the other side, additional mode conversion must take place so that all boundary conditions are satisfied. If, as was true in these measurements, the receiving wedge is similar to the transmitting wedge, the receiver will be most sensitive to that mode which was originally transmitted. The energy at the receiving wedge which is effective in producing an output voltage will be approximately equal to the energy which was imparted to the plate from the transmitting wedge, less the energy which was reflected back down the plate in various modes, less the energy which arrives at the receiving wedge in modes having different phase velocities than the transmitted mode. Since the length of the pulse in the plate is much greater than the width of the saw cut, a standing wave is

TABLE II

INSERTION LOSS IN DECIBELS OF VARIOUS DEPTHS OF SAW CUTS (SAW CUTS FACING UP)

Frequency, Mc angle mode	2.93 40° 1–S	9.15 40° 2–S	10.7 27.5° 3–S	8.93 20° 4–S–A	9.55 30° 3–A	9.50 20° 4–A	10.73 15° 5–A
Depth of cut, inches							
0.003	0.6	6.8	3.9	12.6	5.2	0.6	7.7
0.006	0.1	15.7	4.9	16.1	8.5	1.9	5.7
0.009	−1.9	2.9	−0.6	10.6	5.4	18.4	*
0.012	4.5	12.4	4.0	22.9	23.9	16.9	*
0.015	6.0	4.9	9.9	10.8	10.5	13.1	9.5
0.018	6.9	13.1	6.8	9.2	5.4	*	8.1
0.021	3.8	12.8	5.8	20.7	11.5	*	14.5

*The pulse was too distorted to measure.

present in the plate beneath the saw cut, and its width is important in determining its insertion loss.

The effect of the orientation of the saw cut on its insertion loss was measured in the following manner: An unslotted reference plate was inserted in the apparatus and the frequency was adjusted for maximum amplitude. A slotted plate was then placed in the apparatus with its slot up and the amount of attenuation required to make the signal level at the input of the oscilloscope equal to 0.18 volts was recorded. The same plate was then turned over and the attenuation adjustment was again made and recorded. Ten measurements were made and averaged. The difference between these two values of attenuation is given in Table III.

TABLE III

EFFECT OF THE ORIENTATION OF TEST PLATES UPON THE TRANSMISSION LOSS OF ULTRASOUND—THE TRANSMISSION LOSS WITH THE SAW CUT DOWN REFERRED TO THE TRANSMISSION LOSS WITH THE SAW CUT UP—IN DECIBELS

Frequency, Mc. angle mode	9.4 40° 2–S	10.3 27.5° 3–S	9.3 30° 3–A	9.15 20° 4–S–A
Depth of cut, inches				
0.003	0.6	0.2	0.7	6.3
0.006	1.8	−1.2	0.5	4.0
0.009	2.0	−0.1	−1.7	− 5.9
0.012	0.6	−0.9	0.7	−13.8
0.015	0.0	0.9	−0.7	1.1
0.018	2.3	1.3	1.6	7.8
0.021	0.9	−0.4	1.1	− 2.6

Differences too great to be attributed to experimental error occurred only in the case of the composite fourth symmetrical and asymmetrical mode. Apparently, more of the energy is carried on one side of the plate than on the other because of cancellation of the motion on one side and reinforcement on the other.

The frequencies at which the various modes were found when the data for Table III were taken differed slightly from the corresponding frequencies for the data of Table II. This was attributed to small differences in the adjustment of the angles of the positioning clamps.

CONCLUSION

In order for a complete set of curves to be computed giving the solutions to the period equation of Lamb waves, it is desirable that the period equation be normalized with respect to shear wave velocity. It would also be desirable to have curves of group velocity vs frequency-thickness product which are normalized with respect to shear wave velocity.

Measurements were made of the group velocity of Lamb waves, and the calculated and experimental values agreed satisfactorily.

Measurements made of the insertion loss of saw cuts in 0.033-inch thick steel plates indicate that no simple relationship exists between the depth of the saw cut and the amount of insertion loss. This agrees with intuitive concepts of reflection at a thickness discontinuity. Mode conversion, which is known to exist at such discontinuities,[16,17] is determined by the necessity for satisfying boundary conditions at the discontinuity. The analogous electromagnetic problem has been discussed by Slater.[18]

Measurements were made of the insertion loss of the saw cuts when they were on the same side of the plate as the transducers and when they were on the side of the plate opposite the transducers. A significant difference was found in a case where the fourth symmetrical and asymmetrical modes were being propagated together. This was probably due to cancellation of particle displacement on one side of the plate and reinforcement of particle displacement on the other side so that more power was transmitted along one side than along the other.

APPENDIX I

DERIVATION OF THE FREQUENCY EQUATION FOR LAMB WAVES

By utilizing Hooke's law and Newton's second law of motion it is possible to derive the equations of motion for a lossless isotropic medium:[19]

$$\begin{aligned}
\rho \frac{\partial^2 u}{\partial t^2} &= (\lambda + \mu) \frac{\partial \Delta}{\partial x} + \mu \nabla^2 u, \\
\rho \frac{\partial^2 v}{\partial t^2} &= (\lambda + \mu) \frac{\partial \Delta}{\partial y} + \mu \nabla^2 v, \qquad (3) \\
\rho \frac{\partial^2 w}{\partial t^2} &= (\lambda + \mu) \frac{\partial \Delta}{\partial z} + \mu \nabla^2 w,
\end{aligned}$$

where

$$\Delta = S_1 + S_2 + S_3,$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

In this two-dimensional problem, the x axis is assumed to be horizontal with its positive direction to the right, and the y axis is assumed to be vertical with its positive direction upward. The plate is assumed to be located so that it is symmetrical about the xz plane. The type of wave motion to be considered is independent of z, *i.e.*, all cross sections of the plate parallel to the xy plane are identical, and there is no component of particle motion parallel to the z axis. Consequently, it is required to satisfy only the first two of equations (3).

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[16] F. A. Firestone and D. S. Ling, "Report on the Propagation of Waves in Plates—Lamb and Rayleigh Waves," Sperry Products Inc., Danbury, Conn., Tech. Rept. 50-6001, p. 12; 1945.

[17] D. C. Worlton, "Lamb Waves at Ultrasonic Frequencies," Hanford Atomic Products Operation, Richland, Wash., Rept. No. HW-60662, pp. 29–30; 1959.

[18] J. C. Slater, "Microwave Transmission," McGraw-Hill Book Co., Inc., New York, N. Y., pp. 168–173; 1942.

[19] W. P. Mason, "Physical Acoustics and the Properties of Solids," D. Van Nostrand Co., Inc., Princeton, N. J., pp. 12–14; 1958.

It is readily verified that the solution to the equations of motion for this two-dimensional case is

$$u = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x}, \tag{4}$$

where ϕ and ψ are auxiliary functions satisfying the equations

$$\rho \frac{\partial^2 \phi}{\partial t^2} = (\lambda + 2\mu)\nabla^2 \phi, \qquad \rho \frac{\partial^2 \psi}{\partial t^2} = \mu \nabla^2 \psi. \tag{5}$$

Since $\sqrt{(\lambda + 2\mu)/\rho} = V_L$ = the velocity of a longitudinal wave in an extended medium and $V_S = \sqrt{\mu/\rho}$ = the velocity of a shear wave in an extended medium, (5) may be written as follows:

$$\frac{\partial^2 \phi}{\partial t^2} = V_L^2 \nabla^2 \phi, \qquad \frac{\partial^2 \psi}{\partial t^2} = V_S^2 \nabla^2 \psi. \tag{6}$$

It is assumed that there is periodicity with respect to both x and t; so that ϕ and ψ may be written,

$$\phi = F(y) e^{i(\omega t - kx)}, \qquad \psi = G(y) e^{i(\omega t - kx)}. \tag{7}$$

It is readily seen that

$$\frac{\partial^2 \phi}{\partial t^2} = -\omega^2 \phi, \tag{8}$$

$$\frac{\partial^2 \phi}{\partial x^2} = -k^2 \phi; \tag{9}$$

so that from (6),

$$\frac{\partial^2 \phi}{\partial y^2} = \left(k^2 - \frac{\omega^2}{V_L^2}\right)\phi,$$

defining

$$\begin{aligned} \alpha &= \sqrt{k^2 - \frac{\omega^2}{V_L^2}}\,; \\ \frac{\partial^2 \phi}{\partial y^2} &= \alpha^2 \phi. \end{aligned} \tag{10}$$

Since $\omega = Vk$,

$$\alpha = k\sqrt{1 - \frac{V^2}{V_L^2}}.$$

Thus, for a given value of phase velocity and wavelength, the manner in which ϕ varies with respect to y consists of a solution to an ordinary differential equation in y.

Similarly,

$$\frac{\partial^2 \psi}{\partial y^2} = \beta^2 \psi, \quad \text{where} \quad \beta = k\sqrt{1 - \frac{V^2}{V_S^2}}. \tag{11}$$

The solutions to (10) and (11) are familiar; so that

$$\begin{aligned} F_{(y)} &= a e^{\alpha y} + b e^{-\alpha y}, \\ G_{(y)} &= c e^{\beta y} + d e^{-\beta y}. \end{aligned} \tag{12}$$

$F_{(y)}$ and $G_{(y)}$ may be written as hyberbolic functions so that

$$\begin{aligned} \phi &= (A \sinh \alpha y + B \cosh \alpha y) e^{i(\omega t - kx)}, \\ \psi &= (C \sinh \beta y + D \cosh \beta y) e^{i(\omega t - kx)}. \end{aligned} \tag{13}$$

To obtain the frequency equation, it is necessary to satisfy the conditions that T_2 and T_6 vanish at the two values of $y = \pm h/2$ where h is the thickness of the plate. T_2 and T_6 are given by[19]

$$T_2 = \lambda \Delta + 2\mu S_2, \quad \text{and} \quad T_6 = \mu S_6.$$

$\Delta \equiv \nabla^2 \phi$ so that from (9) and (10),

$$\Delta = (\alpha^2 - k^2)\phi.$$

$$S_2 = \frac{\partial v}{\partial y} = \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x\, \partial y} = \alpha^2 \phi - \frac{\partial^2 \psi}{\partial x\, \partial y}.$$

$$\begin{aligned} T_2 = {} & [\lambda(\alpha^2 - k^2) A \sinh \alpha y + \lambda(\alpha^2 - k^2) B \cosh \alpha y \\ & + 2\mu\alpha^2 A \sinh \alpha y + 2\mu\alpha^2 B \cosh \alpha y + 2ikC\beta\mu \cosh \beta y \\ & + 2ikD\beta\mu \sinh \beta y] e^{i(\omega t - kx)}. \end{aligned}$$

Collecting terms and noting that

$$-\lambda k^2 + \lambda \alpha^2 + 2\mu\alpha^2 \equiv \mu(k^2 + \beta^2),$$

we have

$$\begin{aligned} T_2 = {} & [A\mu(k^2 + \beta^2) \sinh \alpha y + B\mu(k^2 + \beta^2) \cosh \alpha y \\ & + C(2ik\beta\mu) \cosh \beta y + D(2ik\beta\mu) \sinh \beta y] e^{i(\omega t - kx)}. \end{aligned} \tag{14}$$

$$T_6 = \mu S_6$$

$$S_6 = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{\partial^2 \phi}{\partial x\, \partial y} - \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \phi}{\partial x\, \partial y} + \frac{\partial^2 \psi}{\partial y^2}.$$

$$\begin{aligned} T_6 = {} & [A(-2ik\alpha\mu) \cosh \alpha y + B(-2ik\alpha\mu) \sinh \alpha y \\ & + C\mu(k^2 + \beta^2) \sinh \beta y \\ & + D\mu(k^2 + \beta^2) \cosh \beta y] e^{i(\omega t - kx)}. \end{aligned} \tag{15}$$

To make the manipulation less cumbersome, let

$$2ik\beta\mu \equiv n, \qquad -2ik\alpha\mu \equiv p, \qquad \mu(k^2 + \beta^2) \equiv q,$$

$$\sinh(\alpha h/2) \equiv s_\alpha, \qquad \sinh(\beta h/2) \equiv s_\beta,$$

$$\cosh(\alpha h/2) \equiv c_\alpha, \qquad \cosh(\beta h/2) \equiv c_\beta.$$

Then, since at $y = \pm h/2$, $T_2 = T_6 = 0$, four simultaneous equations in the four unknowns A, B, C, and D are obtained:[20]

$$\begin{bmatrix} q s_\alpha & q c_\alpha & n c_\beta & n s_\beta \\ p c_\alpha & p s_\alpha & q s_\beta & q c_\beta \\ -q s_\alpha & q c_\alpha & n c_\beta & -n s_\beta \\ p c_\alpha & -p s_\alpha & -q s_\beta & q c_\beta \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{16}$$

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[20] W. M. Ewing, *et. al.*, *op. cit.*, p. 282.

If B and D and the second and fourth columns of the square matrix are interchanged, the system may be manipulated so that it appears as follows:

$$\begin{bmatrix} qs_\alpha & ns_\beta & 0 & 0 \\ pc_\alpha & qc_\beta & 0 & 0 \\ 0 & 0 & nc_\beta & qc_\alpha \\ 0 & 0 & qs_\beta & ps_\alpha \end{bmatrix} \begin{bmatrix} A \\ D \\ C \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{17}$$

If this system is to have a nontrivial solution, it is necessary that the system determinant vanish,

$$\begin{vmatrix} qs_\alpha & ns_\beta \\ pc_\alpha & qc_\beta \end{vmatrix} \cdot \begin{vmatrix} nc_\beta & qc_\alpha \\ qs_\beta & ps_\alpha \end{vmatrix} = 0. \tag{18}$$

The system determinant will therefore be equal to zero if either of these second-order determinants is zero; furthermore, it is clear that any solution of either of the second-order systems,

$$\begin{bmatrix} qs_\alpha & ns_\beta \\ pc_\alpha & qc_\beta \end{bmatrix} \begin{bmatrix} A \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} nc_\beta & qc_\alpha \\ qs_\beta & ps_\alpha \end{bmatrix} \begin{bmatrix} C \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{19}$$

is a solution of the fourth-order system and furthermore that all combinations of separate solutions to these two second order systems are solutions of the original fourth order system. Therefore, it is possible to separate the original fourth-order system into two independent second-order systems. The two systems are analyzed by assuming first that $A = D = 0$ and deriving the resulting period equations for the remaining second-order system. Thus,

$$\begin{aligned} \phi &= B(\cosh \alpha y)e^{i(\omega t - kx)} \\ \psi &= C(\sinh \beta y)e^{i(\omega t - kx)}, \end{aligned} \tag{20}$$

which implies motion which is symmetrical with respect to the medial plane of the plate, and consequently these vibrations are referred to as the symmetrical modes; whereas, if it is assumed that $B = C = 0$ and the other period equation is solved the functions are

$$\begin{aligned} \phi &= A(\sinh \alpha y)e^{i(\omega t - kx)}, \\ \psi &= D(\cosh \beta y)e^{i(\omega t - kx)}, \end{aligned} \tag{21}$$

Since these functions define motions which are anti-symmetrical about the medial plane, such motions are referred to as the anti-symmetrical modes.

The frequency equation for the symmetrical case is accordingly

$$\begin{vmatrix} nc_\beta & qc_\alpha \\ qs_\beta & ps_\alpha \end{vmatrix} = 0,$$

or

$$npc_\beta s_\alpha = q^2 c_\alpha s_\beta,$$

$$\frac{c_\beta s_\alpha}{c_\alpha s_\beta} = \frac{q^2}{np}.$$

Substituting the original expressions back into the frequency equation, we obtain

$$\frac{\tanh (\alpha h/2)}{\tanh (\beta h/2)} = \frac{(k^2 + \beta^2)^2}{4k^2 \alpha \beta}. \tag{22}$$

Similarly, the frequency equation for the anti-symmetrical case is

$$\frac{\tanh (\alpha h/2)}{\tanh (\beta h/2)} = \frac{4k^2 \alpha \beta}{(k^2 + \beta^2)^2}. \tag{23}$$

APPENDIX II

DERIVATION OF A FORMULA FOR GROUP VELOCITY IN TERMS OF PHASE VELOCITY AND FREQUENCY THICKNESS PRODUCT

It may be shown that the group velocity is equal to the derivative of angular frequency with respect to wave number.[21]

$$V_g = \frac{d\omega}{dk}. \tag{24}$$

Since $\omega = Vk$, $V_g = V + kdV/dk$, and since $dV/df = (dV/dk) \cdot (dk/df)$,

$$\frac{dV}{dk} = \frac{-\dfrac{dV}{df}}{\dfrac{k}{V}\left[\dfrac{dV}{df} - \dfrac{V}{f}\right]} = \frac{-V}{k\left[1 - \dfrac{V}{f\dfrac{dV}{df}}\right]}$$

Therefore,

$$V_g = V\left[1 - \frac{1}{1 - \dfrac{V}{(fh)\dfrac{dV}{d(fh)}}}\right]. \tag{25}$$

To normalize this expression with respect to shear wave velocity, let $V_{gn} = V_g/V_S$, and since $V_n = V/V_S$ and $(fh)_n = fh/V_S$, (25) may be written

$$V_{gn} = V_n\left[1 - \frac{1}{1 - \dfrac{V_n}{(fh)_n \dfrac{dV_n}{d(fh)_n}}}\right].$$

This is the equation used to compute the normalized curves of group velocity vs frequency-thickness product shown in Fig. 2.

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[21] S. Goldman, "Frequency Analysis, Modulation and Noise," McGraw-Hill Book Co., Inc., New York, N. Y., p. 117; 1948.

APPENDIX III

LIST OF NOTATION

a, b, c, d = constants

A, B, C, D = constants

c_α = cosh $(\alpha h/2)$

c_β = cosh $(\beta h/2)$

f = frequency

$(fh)_n$ = normalized frequency-thickness product

$= \dfrac{fh}{V_S}$

$F_{(y)}$ = amplitude of ϕ

$G_{(y)}$ = amplitude of ψ

h = plate thickness

i = complex operator

$= \sqrt{-1}$

k = wave number

$= \dfrac{2\pi f}{V}$

$n = 2ik\beta\mu$

$p = -2ik\alpha\mu$

$q = \mu(k^2 + \beta^2)$

S_1 = tensional strain along the x axis

$= \dfrac{\partial u}{\partial x}$

S_2 = tensional strain along the y axis

$= \dfrac{\partial v}{\partial y}$

S_3 = tensional strain along the z axis

$= \dfrac{\partial w}{\partial z}$

S_6 = shearing strain about the z axis

$= \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}$

s_α = sinh $(\alpha h/2)$

s_β = sinh $(\beta h/2)$

t = time

T_2 = tensional stress parallel to the y axis

T_6 = shear stress about the z axis and parallel to the x axis

u = displacement parallel to the x axis of an elemental cube of volume $dx\,dy\,dz$

v = displacement parallel to the y axis of an elemental cube of volume $dx\,dy\,dz$

V = Lamb wave phase velocity

V_1 = velocity of an incident longitudinal wave

V_g = Lamb wave group velocity

V_{gn} = normalized group velocity

$= V_g/V_S$

V_L = velocity of longitudinal waves in an extended medium of the same elasticity and density as the plate.

$V_n = V/V_S$ = normalized phase velocity

V_S = velocity of shear waves in an extended medium of the same elasticity and density as the plate

w = displacement parallel to the z axis of an elemental cube of volume $dx\,dy\,dz$

x = coordinate in the direction of propagation

y = coordinate normal to the faces of the plate

z = coordinate parallel to the faces of the plate and normal to the direction of propagation

$$\alpha = k\sqrt{1 - \frac{V^2}{V_L^2}}$$

$$\beta = k\sqrt{1 - \frac{V^2}{V_S^2}}$$

Δ = dilation $= S_1 + S_2 + S_3$

θ = angle of incidence, measured to the normal to the faces of the plate, of an incident longitudinal wave

λ = Lamé's constant

μ = shear modulus

ρ = density

ϕ = auxiliary function for dilation

ψ = auxiliary function for rotation

ω = angular frequency

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

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